

On the stability of vortex-plane solitons: The solution of the problem of Josephson-vortex structure in layered superconductors and stacked junctions

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By determining the type of all stationary points of the Gibbs free energy functional for layered superconductors in parallel magnetic fields, we establish the classification of all solutions to coupled static sine-Gordon equations for the phase differences with respect to their stability. We prove that the only minimizers of the free energy are the Meissner solution (the "vacuum" state) and soliton vortex-plane solutions [S. V. Kuplevakhsky, Phys. Rev. B **60**, 7496 (1999); *ibid.* **63**, 054508 (2001); cond-mat/0202293]. They are the actual equilibrium field configurations. We present a topological classification of these solutions. In contrast, previously proposed non-soliton configurations ("isolated fluxons", "triangular Josephson-vortex lattices", etc.) are absolutely unstable and unobservable: They are nothing but saddle points of the Gibbs free-energy functional and are not even stationary points of the Helmholtz free-energy functional (obtained from the former by a Legendre transformation). (Physically, non-soliton configurations violate conservation laws for the current and the flux.) The obtained results allow us to explain dynamic stability of vortex planes, noticed in numerical simulations, and to provide a unified interpretation of the available experimental data. We hope that the paper will stimulate interest in the subject of specialists in different fields of physics and in applied mathematics.

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I. INTRODUCTION

In this paper, we present the solution of the problem of equilibrium vortex structure in layered superconductors and stacked Josephson junctions in the presence of a parallel, static, homogeneous external magnetic field H and provide a unified interpretation of the available experimental data. Our approach consists in a rigorous mathematical analysis of the stability of all types of flux configurations, proposed in the literature, by means of exact variational methods for microscopic free-energy functionals.^{1,2,3}

Within the framework of these methods, we have previously obtained a complete classification of all possible static soliton solutions to coupled sine-Gordon (SG) equations for phase differences both in infinite [$N = \infty$, N is the number of superconducting(S) layers] layered superconductors^{1,2} and finite ($N < \infty$) Josephson-junction stacks³ for $H > 0$. Based on the fundamental argument of soliton physics^{4,5,6,7,8,9} that topological solitons in nonlinear field theories are minimizers of the energy functionals (free-energy functionals in our case),¹⁰ we have identified these solutions with equilibrium Josephson-vortex configurations. Their magnetic field has symmetry typical of plane defects, hence the term "vortex planes". Physically, a vortex plane can be regarded as a bound state of interlayer vortices (one vortex per each insulating layer in the plane). In contrast to a deep-rooted belief¹¹ in an "analogy" with Abrikosov vortices in continuum type-II superconductors, the SG equations for $H > 0$ do not admit static soliton solutions that can be identified with an "isolated Josephson vortex" or a "triangular Josephson-vortex lattice".

Unfortunately, a wide-spread misunderstanding of the fact that equilibrium Josephson vortices are nothing but static soliton solutions to the SG equations incurred a misunderstanding of the exact mathematical results of Refs. 1,2,3. In the critical comment, Ref. 12, V. M. Krasnov and L. N. Bulaevskii "disprove" the conclusions of Refs. 1,2 by claiming (in contradiction to the fundamentals of soliton physics) that vortex-plane solitons "maximize the free energy".¹³ According to Ref. 12, the "instability" of vortex planes is "similar to the instability of the laminar solution¹⁴ for type-II superconductors". [Alternating superconducting and normal layers, envisaged by the laminar model,¹⁴ have nothing to do with soliton physics and do not possess the property of topological stability: see a proof in subsection V.B of the present paper.] V. M. Krasnov and L. N. Bulaevskii insist on hypothetical "isolated fluxons", allegedly, having "lower energy" for $H > 0$ and characterized by a "much smaller length scale". No exact mathematical definition of the "isolated fluxons" is given. We have not found any definition in the original papers by the critics of vortex planes, either: For example, it is claimed in Refs. 15,16 that Josephson vortices "do not exist" in single Josephson junctions with $W \ll 2\lambda_J$ (W is the junction width, λ_J is the Josephson length). However, an exact, closed-form analytical solution^{3,17} to the single static SG equation, appropriate for this case, clearly demonstrates the

existence of phase-difference solitons for arbitrarily small W , provided the external field H is sufficiently high. [As shown in Refs. 1,2, exactly these solitons account for the well-known Fraunhofer pattern of the critical Josephson current $I_c(H)$ for $W \ll 2\lambda_J$.]

Furthermore, the manuscript Ref. 3, although submitted twice to Physical Review B (November 2000, August 2001), is still not published. In particular, one of the referees disputed the conclusions of Ref. 3, because, in his opinion, the soliton boundary conditions^{5,6,7,8,9} employed therein "overdetermined" the problem of the classification of equilibrium Josephson-vortex configurations. He argued that certain numerical simulations for the SG equations had demonstrated, aside from vortex-plane solitons, the existence of "single-vortex" solutions. According to the referee, these solutions had "lower free energy" than the vortex planes for given H . As in Ref. 12, no exact definition of such solutions was given.

It should be emphasized that the idea of an "analogy" between the Josephson-vortex structure in layered superconductors and the Abrikosov-vortex structure in continuum type-II superconductors was not supported in Ref. 11 and subsequent publications^{18,19,20,21} by any serious mathematical arguments.^{1,2,22} Neither was it confirmed by direct experimental observations of the equilibrium Josephson-vortex structure in artificial stacked junctions at $H > 0$.^{23,24} Unfortunately, most theoretical efforts were constrained by the idea of an "analogy", hence the use of mathematically ill-formulated methods, such as, e.g., a "continuum-limit approximation".^{11,18,19,20,21} For instance, the exact SG equations for the phase differences were not even derived in Refs. 11,18, concerned with a "single Josephson vortex" at $H > 0$. (As shown by Farid,²² equations of Ref. 18 have no physical solution.) Since the problem of the stability of the proposed "vortex configurations" (i.e., whether they are actual points of minima of the free-energy functionals) required the use of rigorous mathematical methods, it was not even posed in Refs. 11,18,19,20,21.

Concerning numerical simulations for the static SG equations,^{16,25} there is an unjustified tendency to identify any kink-type feature of the phase difference with a "Josephson vortex", without any analysis of its stability. In contrast to the exact analytical methods of Refs. 1,2,3, the numerical approach does not provide any means to establish a full set of necessary and sufficient conditions of the minimum of the free-energy functionals. Typically,²⁵ numerical simulations start with an incorrectly formulated (both mathematically and physically) boundary value problem that does not meet the criterion of uniqueness.²⁶

To close the issue of the equilibrium Josephson-vortex structure in layered superconductors and stacked junctions, we determine analytically (by means of exact methods of the calculus of variations²⁷ and soliton physics^{4,6,7,8,9,28}) the type of all stationary points of the exact microscopic Gibbs free-energy functional,³ generating the static SG equations. Our consideration applies to an arbitrary number of superconducting layers N , including the cases $N = 2$ (a single junction) and $N \rightarrow \infty$ (an infinite layered superconductor). As a result, we obtain a complete classification of all nontrivial solutions to the SG equations, considered in the literature (both analytically and numerically), with respect to their stability. As could be expected from the general arguments of soliton physics, the only minimizers of the free-energy functional are the Meissner solution (the "vacuum" state) and soliton vortex-plane solutions.^{1,2,3} The latter represent the actual equilibrium Josephson-vortex configurations for $H > 0$. In contrast, non-soliton configurations (e.g., "isolated fluxons", "triangular Josephson-vortex lattices", etc.) are absolutely unstable: They are nothing but saddle points of the Gibbs free-energy functional and are not even stationary points of the Helmholtz free-energy functional (obtained from the former by a Legendre transformation). [Physically, non-soliton solutions violate conservation laws for the current and the flux, as was first noticed in the case of an infinite layered superconductor ($N = \infty$) in Refs. 1,2.]

In section II, we clarify a relationship between the correct formulation of the boundary value problem to the SG equations and a full set of necessary and sufficient conditions of the minimum of the Gibbs free-energy functional. The proof of the stability of the Meissner solution and vortex-plane solitons is given in sections III, IV. In section III, we establish that the sufficient condition of the minimum of the Gibbs free-energy functional consists in the vanishing of the surface variation of the corresponding Helmholtz free-energy functional, which, in turn, yields conservation laws for the flux and the intralayer current. In subsection IV.A, we derive the soliton boundary conditions directly from the conservation law for the flux, which provides the sought proof of the stability of the Meissner solution and vortex-plane solitons. The main physical and mathematical properties of these solutions are discussed in subsection IV.B. In subsection IV.C, we analyze the obtained results from the point of view of general theory of topological defects^{4,5,6,7,8,9,29} and explain dynamic stability of vortex planes, established in numerical simulations.^{30,31,32}

Some important physical and mathematical issues, related to the main results of the paper, are discussed in section V. In subsection V.A, we present a rigorous analytical description of unstable solutions ("isolated fluxons", "triangular Josephson-vortex lattices", etc.), proposed in previous theoretical publications and numerical simulations. In subsection V.B, we draw a comparison with the Abrikosov vortices in type-II superconductors. In subsection V.C, we analyze the available experimental data from the point of view of the stable vortex-plane configurations.

In section VI, we summarize the obtained results, systematize our criticism of previous approaches and make some concluding remarks. In Appendix A, we give a list of mathematical formulas, relevant to the main text. In Appendix B, we establish a relationship to the variational principle of Refs. 1,2 for infinite layered superconductors.

Throughout the paper, we adhere to the dimensionless notation of Ref. 3. The geometry of the problem is that of figures 1, 2 in Ref. 3: The superconductor occupies the region $[0 \leq x \leq N-1] \times [-L \leq y \leq L] \times (-\infty < z < \infty)$, where $2L = W$; the layering axis (the c -axis) is x ; the axis y is along the layers; the external magnetic field is along the axis z : $\mathbf{H} = (0, 0, H \geq 0)$. The phase difference between two successive S-layers is denoted as $\phi_n \equiv \varphi_n - \varphi_{n-1}$ ($\phi_0 = \phi_N \equiv 0$), with $n = 0, 1, \dots, N-1$ being the S-layer number. The c -axis external current is not considered: $I = 0$.

II. THE FORMULATION OF THE PROBLEM

In Ref. 3, we have derived the coupled static SG equations for the phase differences ϕ_n by minimizing the exact microscopic Gibbs free-energy functional² $\Omega[f_n, \phi_n, \mathbf{A}; H]$ with respect to f_n and \mathbf{A} (f_n is the reduced modulus of the order parameter in the n -th S-layer, \mathbf{A} is the vector potential). In the limit $r(T) \ll 1$, $H \ll H_{c2}$ [$r(T)$ is the parameter of the interlayer coupling, H_{c2} is the upper critical field], when $f_n = 1$, the SG equations appear as solubility conditions for the Maxwell equations in the gauge

$$\mathbf{A} = [0, A(x, y), 0]. \quad (1)$$

The SG equations read:³

$$\frac{d^2 \phi_n(y)}{dy^2} = \frac{1}{\epsilon^2} \sum_{m=1}^{N-1} G^{-1}(n, m) \sin \phi_m(y), \quad n = 1, \dots, N-1; \quad (2)$$

where $G^{-1}(n, m)$ is a Jacobian matrix with elements $G^{-1}(n, n) = 2 + \epsilon^2$ ($n = 1, \dots, N-1$), $G^{-1}(n+1, n) = G^{-1}(n, n+1) = -1$ ($n = 1, \dots, N-2$), and $G^{-1}(n, m) = 0$ for $|n-m| > 1$.

The requirement that the local field $H_n(y)$ ($n-1 \leq x < n$, $n = 1, \dots, N-1$) be equal to the applied one at $y = \pm L$ has led to the conditions

$$\frac{d\phi_n}{dy}(-L) = \frac{d\phi_n}{dy}(L), \quad n = 1, \dots, N-1; \quad (3)$$

$$\frac{d\phi_n}{dy}(-L) = \frac{d\phi_{n+1}}{dy}(-L) \equiv \frac{d\phi}{dy}(-L) \geq 0, \quad n = 1, \dots, N-2. \quad (4)$$

[The condition $\frac{d\phi}{dy}(-L) \geq 0$ merely reflects the fact that the local field is parallel to the applied one $H \geq 0$.] From the requirement that the local field be equal to the applied one at $x = 0$, $x = N-1$, we have obtained

$$\phi_n(y) = \phi_{N-n}(y), \quad n = 1, \dots, N-1. \quad (5)$$

Equations (2) and boundary conditions (3) are satisfied by functions of the type

$$\phi_n(y) = -\phi_n(-y) + 2\pi Z_n, \quad (6)$$

where the constants Z_n can be arbitrarily chosen from the set $0, \pm 1, \pm 2, \dots$. As is pointed out in Ref. 3, the fixation of the constants Z_n requires imposition of boundary conditions on ϕ_n at $y = \pm L$. Based on general arguments of soliton physics that soliton solutions are minimizers of corresponding energy functionals, we have imposed in Ref. 3 standard soliton boundary conditions on ϕ_n .

As it has turned out, the only possible solutions, compatible with the requirement (4), are the Meissner solution and the soliton vortex-plane solutions, for which (5) is satisfied automatically and

$$Z_n = Z_{n+1} \equiv N_v, \quad n = 1, \dots, N-2; \quad N_v = 0, 1, 2, \dots \quad (7)$$

in (6), with $N_v = 0$ representing the topologically trivial Meissner solution. Given that³

$$H_n(y) = H[G(n, 1) + G(n, N-1)] + \frac{\epsilon^2}{2} \sum_{m=1}^{N-1} G(n, m) \frac{d\phi_m(y)}{dy}, \quad (8)$$

the field $H = \bar{H}$, corresponding to a concrete configuration $\{\bar{\phi}_n\}$ with $N_v = \bar{N}_v$, is determined by

$$\frac{d\bar{\phi}}{dy}(-L) = 2\bar{H}. \quad (9)$$

Note that the first, phase-independent, term in (8) is a contribution of the field penetrating through the boundaries $x = 0$, $x = N - 1$, and the second term is a contribution of the field penetrating through the boundaries $y = \pm L$. The matrix $G(n, m)$ is the inverse of $G^{-1}(n, m)$: for its properties, see Appendix A.

In contrast to the above exact variational method, numerical simulations^{16,25} for (2) start with the imposition of the boundary conditions

$$\frac{d\phi_n}{dy}(\pm L) = 2H, \quad (10)$$

without any regard to sufficient conditions of the minimum of the Gibbs free-energy functional. Such an approach is based on an erroneous belief that all solutions to (2), (10) minimize the free-energy functional. (Manifestations of this belief are the naive "energy arguments" of Ref. 12, appealing to "a difference in the length scales", and calculations by means of combinatorics^{16,25} of the "number of quasi-equilibrium fluxon modes".) However, conditions (10) do not specify any boundary value problem for (2): By virtue of the symmetry relations (6), the imposition of the boundary condition on $\frac{d\phi_n}{dy}$ at $y = -L$ automatically ensures the fulfillment of the same boundary condition at $y = L$, whereas the constants Z_n remain undetermined. Thus, the "boundary value problem" (10) does not satisfy the criterion of uniqueness,²⁶ which is a sign of the presence of unphysical (i.e., unobservable) solutions.

The existence of redundant solutions to (2), (10) is already clear for physical reasons: This "boundary value problem" does not take any account of the necessity to ensure the continuity of the local field at the boundaries $x = 0$, $x = N - 1$. To understand at a rigorous mathematical level where the unphysical solutions come from, we have to consider all stationary points of the generating Gibbs free-energy functional, rewritten via ϕ_n and $\frac{d\phi_n}{dy}$.³ In this way, we will derive a full set of necessary and sufficient conditions of the minimum directly from the variational principle, obtain an independent proof of the fact that the Meissner solution and the soliton vortex-plane solutions^{1,2,3} are the unique minimizers of the problem and establish the character of the instability of unphysical non-soliton solutions. In our consideration, we will employ the first integral of (2) that, taking account of (10), has the form³

$$\begin{aligned} & \sum_{n=1}^{N-1} \cos \phi_n(y) + \frac{\epsilon^2}{2} \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n, m) \frac{d\phi_n(y)}{dy} \frac{d\phi_m(y)}{dy} \\ &= \frac{2H^2}{H_s^2} (N-1) + \sum_{n=1}^{N-1} \cos \phi_n(L), \end{aligned} \quad (11)$$

where H_s is the superheating (penetration) field of a semiinfinite ($0 \leq y < +\infty$) Josephson-junction stack, given by Eq. (A8).

III. THE NECESSARY AND SUFFICIENT CONDITIONS OF THE MINIMUM OF THE GIBBS FREE-ENERGY FUNCTIONAL

The generating Gibbs free-energy functional for the SG equations (2) has the form³

$$\Omega \left[\phi_n, \frac{d\phi_n}{dy}; H \right] = F \left[\phi_n, \frac{d\phi_n}{dy}; H \right] - 4Hr(T) \sum_{n=1}^{N-1} \Phi_n \frac{\phi_n(L) - \phi_n(-L)}{2\pi}, \quad (12)$$

$$F \left[\phi_n, \frac{d\phi_n}{dy}; H \right]$$

$$\begin{aligned} &= r(T) \left[\frac{2H^2}{H_s^2} W(N-1) + \frac{\epsilon^2}{2} \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n, m) \int_{-L}^L dy \frac{d\phi_n(y)}{dy} \frac{d\phi_m(y)}{dy} \right. \\ &\quad \left. + \sum_{n=1}^{N-1} \int_{-L}^L dy [1 - \cos \phi_n(y)] \right], \end{aligned} \quad (13)$$

$$\Phi_n = \pi [1 - G(n, 1) - G(n, N - 1)]. \quad (14)$$

Note that the functional (12) is measured from the condensation energy $\Omega_0 = -\frac{NW}{2}$. Moreover, the energy of the external field in the absence of the sample is subtracted.³³

Our task is to establish the necessary and sufficient conditions of the minimum of (12):

$$\begin{aligned} & \Delta\Omega \left[\phi_n, \frac{d\phi_n}{dy}; H \right] \\ &= \Omega \left[\phi_n + \delta\phi_n, \frac{d\phi_n}{dy} + \frac{d\delta\phi_n}{dy}; H \right] - \Omega \left[\phi_n, \frac{d\phi_n}{dy}; H \right] \geq 0 \end{aligned} \quad (15)$$

First, we observe that, in contrast to (12), the functional (13) is positive definite

$$F \left[\phi_n, \frac{d\phi_n}{dy}; H \right] \geq 0, \quad (16)$$

since the matrix $G(n, m)$ is positive definite: see Appendix A. The absolute minimum of (13) is achieved for $H = 0$, $\phi_n \equiv 0$ ($n = 1, 2, \dots, N - 1$). Hence, the functional (13) necessarily has minima for any $H \geq 0$. The first variations of (12), (13) are

$$\delta\Omega \left[\phi_n, \frac{d\phi_n}{dy}; H \right] = \delta F \left[\phi_n, \frac{d\phi_n}{dy}; H \right] - 4Hr(T) \sum_{n=1}^{N-1} \Phi_n \frac{\delta\phi_n(L) - \delta\phi_n(-L)}{2\pi}, \quad (17)$$

$$\begin{aligned} & \delta F \left[\phi_n, \frac{d\phi_n}{dy}; H \right] \\ &= r(T) \sum_{n=1}^{N-1} \int_{-L}^L dy \left[\sin \phi_n(y) - \epsilon^2 \sum_{m=1}^{N-1} G(n, m) \frac{d^2 \phi_m(y)}{dy^2} \right] \delta\phi_n(y) \\ &+ 2 \frac{d\phi}{dy}(-L) r(T) \sum_{n=1}^{N-1} \Phi_n \frac{\delta\phi_n(L) - \delta\phi_n(-L)}{2\pi}. \end{aligned} \quad (18)$$

The requirement that the volume variation in (18) vanish yields the SG equations (2), as expected. Of special interest to us are surface variations, i.e., the last terms in (17), (18): The requirement that these variations vanish determines boundary conditions to (1). (For a very clear discussion of the relationship between the surface variation and boundary conditions, see Ref. 27, section II.15.) In the derivation of the surface variation in (18), we have used conditions (3), (4) that the local field be continuous at the boundaries $y = \pm L$. The requirement of the continuity of the local field at the boundaries $x = 0$, $x = N - 1$ has not been so far employed. [Recall our remark in section II that the disregard of this requirement is the reason for unphysical solutions to (1), (10).]

If we simply enforce the conditions (10), the surface variations in (17) and (18) cancel out: Thus, all solutions to (2), (10) are stationary points of the Gibbs free-energy functional (12). However, under (10), the surface variation in (18) does not vanish. Therefore, not all solutions to (2), (10) are stationary points of (13). We have to examine conditions of the stationarity of (13) in more detail.

The requirement that the local field is fixed at the boundaries $x = 0$, $x = N - 1$ is equivalent to the requirement that the vector potential \mathbf{A} is fixed at $x = 0$, $x = N - 1$. Consider now the total flux Φ . In the gauge (1), we have

$$\Phi = \int_{-L}^L dy [A(N - 1, y) - A(0, y)]. \quad (19)$$

On the other hand,

$$\Phi = \sum_{n=1}^{N-1} \int_{-L}^L dy H_n(y) = HW(N - 1) \frac{H_s^2 - 1}{H_s^2} + \sum_{n=1}^{N-1} \Phi_n \frac{\phi_n(L) - \phi_n(-L)}{2\pi}, \quad (20)$$

where the first term is the flux penetrating through the boundaries $x = 0$, $x = N - 1$, and the second term is the flux penetrating through the boundaries $y = \pm L$. (We will call it the "Josephson flux", Φ_J .) Given that

$$\delta A(0, y) = \delta A(N - 1, y) = 0, \quad (21)$$

we have

$$\delta\Phi = \delta\Phi_J = \sum_{n=1}^{N-1} \Phi_n \frac{1}{2\pi} \int_{-L}^L dy \frac{d\delta\phi_n(y)}{dy} = \sum_{n=1}^{N-1} \Phi_n \frac{\delta\phi_n(L) - \delta\phi_n(-L)}{2\pi} = 0. \quad (22)$$

Thus, the continuity of the field at $x = 0$, $x = N - 1$ imposes a constraint on the variations:

$$\Phi_J = \sum_{n=1}^{N-1} \Phi_n \frac{1}{2\pi} \int_{-L}^L dy \frac{d\phi_n(y)}{dy} = \text{const} \geq 0. \quad (23)$$

The result (23) is exactly what had to be expected:³³ By virtue of the Meissner effect, the flux Φ_J (and, of course, Φ) is stable against any small perturbations, represented by variations $\phi_n(y) \rightarrow \phi_n(y) + \delta\phi_n(y)$. Equivalent forms of (22), (23) are

$$\delta\phi_n(L) = \delta\phi_n(-L), \quad (24)$$

$$\phi_n(L) - \phi_n(-L) = c_n = \text{const} \geq 0. \quad (25)$$

Note that the existence of conserved physical quantities of the type of Φ_J is a precursor to the existence of soliton solutions in nonlinear field theories.^{4,5,6,7,8,9,28} In Appendix B, we establish a relationship between the conservation of Φ_J and the conservation of the intralayer current, which, in turn, establishes a relationship to the variational principle for infinite ($N = \infty$) layered superconductors.^{1,2}

What will be shown now is that all the stationary points of (13) are the unique minimizers of both (12) and (13). First, we notice that high-order variations of (12) and (13) coincide: $\delta^n F = \delta^n \Omega$, $n \geq 2$. Thus, all the minimizers of (13) are minimizers of (12). On the other hand, the minimizers of (12) obeying (23) are minimizers of (13): From the condition of the minimum (15), we get

$$\begin{aligned} & F \left[\phi_n + \delta\phi_n, \frac{d\phi_n}{dy} + \frac{d\delta\phi_n}{dy}; H \right] - F \left[\phi_n, \frac{d\phi_n}{dy}; H \right] - \delta\Phi_J \\ &= F \left[\phi_n + \delta\phi_n, \frac{d\phi_n}{dy} + \frac{d\delta\phi_n}{dy}; H \right] - F \left[\phi_n, \frac{d\phi_n}{dy}; H \right] \geq 0. \end{aligned} \quad (26)$$

[Physically, this fact means the equivalence of the description in terms of the Gibbs free energy and the Helmholtz free energy: Because of (23), the functional $F \left[\phi_n, \frac{d\phi_n}{dy}; \Phi_J \right] \equiv F \left[\phi_n, \frac{d\phi_n}{dy}; 0 \right]$ can be regarded as the Helmholtz free-energy functional.] Using the standard technique,²⁸ it is straightforward to prove that all stationary points of (12), obeying (23), are minimizers of (12) [and, thus, of (13)]. Indeed, let $\{\bar{\phi}_n\}$ be the stationary point for $\Phi_J = \bar{\Phi}_J$ and corresponding $H = \bar{H}$. In the vicinity of $\{\bar{\phi}_n\}$, i.e., for $\phi_n = \bar{\phi}_n + \delta\bar{\phi}_n$, we have the following estimate:

$$\begin{aligned} & \Omega \left[\phi_n, \frac{d\phi_n}{dy}; \bar{H} \right] \\ & \geq r(T) \left[\frac{\epsilon^2}{2} \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n, m) \int_{-L}^L dy \frac{d\phi_n(y)}{dy} \frac{d\phi_m(y)}{dy} - 4\bar{H} \sum_{n=1}^{N-1} \Phi_n \frac{1}{2\pi} \int_{-L}^L dy \frac{d\phi_n(y)}{dy} \right] \\ & \geq -4\bar{H}r(T) \bar{\Phi}_J. \end{aligned} \quad (27)$$

Inequalities (27) show that Ω has a lower bound in the vicinity of any stationary point $\{\bar{\phi}_n\}$, obeying (23); hence, $\{\bar{\phi}_n\}$ is a minimizer of Ω and F , Q.E.D.

To strengthen (27), we minimize the right-hand side of the first inequality with respect to $\frac{d\phi_n}{dy}$, obtaining

$$\epsilon^2 \sum_{m=1}^{N-1} G(n, m) \left[\frac{d\phi_m}{dy} \right]_{\min} = \frac{2\bar{H}}{\pi} \Phi_n, \quad (28)$$

$$\Omega \left[\phi_n, \frac{d\phi_n}{dy}; \bar{H} \right] \geq -2\bar{H}r(T) \bar{\Phi}_J. \quad (29)$$

Taking into account that $\{\bar{\phi}_n\}$ is a solution of (2), making use of (11) and (28), we get:

$$\begin{aligned} \Omega \left[\phi_n, \frac{d\phi_n}{dy}; \bar{H} \right] &\geq \Omega \left[\bar{\phi}_n, \frac{d\bar{\phi}_n}{dy}; \bar{H} \right] \\ &= r(T) \left[W \sum_{n=1}^{N-1} [1 - \cos \bar{\phi}_n(L)] + \epsilon^2 \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n, m) \int_{-L}^L dy \frac{d\bar{\phi}_n(y)}{dy} \frac{d\bar{\phi}_m(y)}{dy} - 4\bar{H}\bar{\Phi}_J \right] \\ &\geq r(T) W \sum_{n=1}^{N-1} [1 - \cos \bar{\phi}_n(L)] \geq 0. \end{aligned} \quad (30)$$

The inequality $\Omega \left(\bar{\phi}_n, \frac{d\bar{\phi}_n}{dy}; \bar{H} \right) \geq 0$ is a manifestation of the Meissner effect and had to be expected from general thermodynamic arguments.³³ Relations (30) immediately yield

$$\begin{aligned} F \left[\phi_n, \frac{d\phi_n}{dy}; \bar{H} \right] &\geq F \left[\bar{\phi}_n, \frac{d\bar{\phi}_n}{dy}; \bar{H} \right] \\ &\geq r(T) W \sum_{n=1}^{N-1} [1 - \cos \bar{\phi}_n(L)] + 4\bar{H}r(T) \bar{\Phi}_J \geq 4\bar{H}r(T) \bar{\Phi}_J. \end{aligned} \quad (31)$$

We want to emphasize that inequalities of the type (29), (31) are typical of soliton physics: They are used to establish the existence and stability of soliton solutions.^{4,6,7,8,9,28}

Note an alternative interpretation of the variational principle

$$\delta\Omega \left[\phi_n, \frac{d\phi_n}{dy}; H \right] = \delta F \left[\phi_n, \frac{d\phi_n}{dy}; 0 \right] - 4Hr(T) \sum_{n=1}^{N-1} \Phi_n \frac{1}{2\pi} \delta \int_{-L}^L dy \frac{d\phi_n(y)}{dy} = 0. \quad (32)$$

The field H in (32) can be considered as a Lagrange multiplier, implying that variation can be performed without any restrictions on $\delta\phi_n(L)$, $\delta\phi_n(-L)$. In this case, the requirement that the surface variation vanish yields conditions (3), (4). Boundary conditions on ϕ_n are uniquely determined by (23): see the next section. The value $H = \bar{H}$ for a concrete minimizer $\{\bar{\phi}_n, \bar{\Phi}_J\}$ should be found from the condition of thermodynamic equilibrium

$$\frac{\partial\Omega \left(\bar{\phi}_n, \frac{d\bar{\phi}_n}{dy}; \bar{H} \right)}{\partial\bar{\Phi}_J} = 0. \quad (33)$$

Indeed, $\Omega \left(\bar{\phi}_n, \frac{d\bar{\phi}_n}{dy}; \bar{H} \right)$ can be written as

$$\Omega \left(\bar{\phi}_n, \frac{d\bar{\phi}_n}{dy}; \bar{H} \right) = r(T) \left[W \sum_{n=1}^{N-1} \left[1 - \cos \bar{\phi}_n(L) - \frac{1}{W} \int_{-L}^L dy \bar{\phi}_n(y) \sin \bar{\phi}_n(y) \right] \right]$$

$$+2 \left[\frac{d\bar{\phi}}{dy}(L) - 2\bar{H} \right] \bar{\Phi}_J, \quad (34)$$

which by virtue of (33) immediately yields (9).

In summary, we have proved the following: The SG equations (1) and relations (10) ensure only the stationarity of the Gibbs free-energy functional (12). The necessary and sufficient conditions of the minimum of both (12) and (13) (which is the Helmholtz free-energy functional for $H = 0$) are given by (3), (4) and the constraint (23). Solutions to (2), (10) that do not obey this constraint are absolutely unstable. The character of this instability can be easily established. Indeed, such solutions are not even stationary points of the Helmholtz free-energy functional, therefore $\delta^2\Omega = \delta^2F$ need not have definite sign. Moreover, the functional (12) is unbounded in the vicinity of these solutions. Thus, they are nothing but saddle points of (12).

IV. THE PROOF OF THE STABILITY OF THE MEISSNER SOLUTION AND VORTEX-PLANE SOLITONS

A. Boundary conditions on ϕ_n

Our aim now is to establish boundary conditions on ϕ_n directly from the constraint (23). Given that the SG equations (2), boundary conditions (3), (4) and the constraint (23) represent a full set of necessary and sufficient conditions of the minimum of the Gibbs and Helmholtz free-energy functionals, we will obtain, in this manner, the sought proof of the stability of the Meissner solution and soliton vortex-plane solutions.

First, we observe that since a minimizer of (12), (13) must ensure the vanishing of both the surface and volume variations in (17) and (18), it should necessarily belong to the class of functions that satisfy (2), (3), (4) and the symmetry relations (6). Thus, the variation of the surface terms in (17) and (18) is performed with respect to trial functions that take only discrete values at $y = 0$:

$$\phi_n(0) = \pi Z_n, \quad (35)$$

where Z_n can be arbitrarily chosen from the set $0, \pm 1, \pm 2, \dots$. These functions can be subdivided into classes parameterized by an $(N - 1)$ -dimensional "vector"

$$\mathbf{Q} = (Z_1, Z_2, \dots, Z_{N-1}). \quad (36)$$

In view of (35), the requirement of the continuity of variations can be met if and only if

$$\delta\phi_n(0) = 0, \quad (37)$$

which means that all the minima of (12), (13) are parameterized by the vector \mathbf{Q} , and the variation of the surface terms in (17) and (18) is performed with respect to trial functions that belong to a certain class (36). Moreover, the symmetry relations (6) imply $\delta\phi_n(L) = -\delta\phi_n(-L)$. Combined with (14), this yields

$$\delta\phi_n(L) = \delta\phi_n(-L) = 0. \quad (38)$$

Now we combine (15) with (6) to obtain

$$\phi_n(L) - \pi Z_n = \frac{c_n}{2} \geq 0, \quad (39)$$

$$-\phi_n(-L) + \pi Z_n = \frac{c_n}{2} \geq 0. \quad (40)$$

Since the inequalities in (39), (40) should hold for any Z_n , including $Z_n = 0$, we get

$$\phi_n(-L) \leq 0, \quad \phi_n(L) \geq 0.$$

Moreover, since for any fixed set $\{c_n\}$ the set $\{\phi_n(\pm L)\}$ must belong to a certain unique class (36),

$$-\pi < \phi_n(-L) \leq 0, \quad (41)$$

$$2\pi Z_n \leq c_n < 2\pi (Z_n + 1), \quad Z_n = \left[\frac{1}{2\pi} \int_{-L}^L dy \frac{d\phi_n(y)}{dy} \right] = 0, 1, 2, \dots, \quad (42)$$

where $[u]$ is the integer part of u .

Given (37), (38), the boundary conditions (41) and

$$\phi_n(0) = \pi Z_n, \quad Z_n = 0, 1, 2, \dots \quad (43)$$

together with (3), (4) determine, in principle, a complete set of conditions for the minimizer of (12), (13). The solution belonging to a certain class (36) [with Z_n as in (43)] first appears when all $\phi_n(-L) = 0$, and $c_n = 2\pi Z_n$ [see the estimates (30), (31): under these conditions, the density of the Josephson energy at the boundary is a minimum]. Thus, we are confronted with the standard^{5,6,7,8,9} soliton boundary value problem

$$\phi_n(-L) = 0, \quad \phi_n(0) = \pi Z_n, \quad Z_n = 0, 1, 2, \dots, \quad (44)$$

plus the boundary conditions (4). To find out what type of minimizers can be realized in reality, we have to solve Eqs. (2).^{3,17} Note that $N - 2$ relations (4) implicitly impose $N - 2$ conditions on $N - 1$ constants Z_n . Therefore, as shown in Refs. 3,17, the only solutions to (2), (44), compatible with (4), are those that satisfy (7), i.e., the Meissner solution and the soliton vortex-plane solutions:

$$\mathbf{Q}_v = (N_v, N_v, \dots, N_v), \quad N_v = \left[\frac{1}{2\pi} \int_{-L}^L dy \frac{d\phi_n(y)}{dy} \right] = 0, 1, 2, \dots \quad (45)$$

The properties of these solutions will be discussed in more detail in what follows. Here we want to brief on the results for $H = 0$.^{3,17} For $H = 0$, $L < \infty$, there are no soliton solutions at all (including the vortex planes), and the only stable solution is the trivial Meissner solution $\phi_1 = \phi_2 = \dots \phi_{N-1} \equiv 0$. The situation changes drastically for $H = 0$, $L = \infty$: The imposition of soliton boundary conditions on $\phi_n(\pm\infty)$ automatically ensures the fulfillment of the boundary conditions $\frac{d\phi_n}{dy}(\pm\infty) = 0$ by virtue of Eqs. (2) themselves and some elementary theorems of mathematical analysis. Aside from the vortex-plane solution with $N_v = 1$ in (45) (where $L = \infty$), we have a variety of soliton solutions (36) with Z_n arbitrarily chosen from the set $0, \pm 1$. The fact that for $H = 0$, $L = \infty$ each ϕ_n can "accommodate" no more than one vortex or antivortex is a generalization of the well-known^{5,6} result for the single static SG equation and can be easily proved by the use of the first integral (11) with the right-hand side equal to $N - 1$.

B. The Meissner solution and soliton vortex-plane solutions

The range of the existence of the solutions, parameterized by (45), is determined by the boundary value problem (41), (43) (with all $Z_n = N_v$) and relation (45):^{3,17}

$$0 \leq H < H_0 \equiv H_{sL}, \quad \text{for } N_v = 0, \quad (46)$$

$$\sqrt{H_{N_v-1}^2 - H_s^2} \leq H < H_{N_v}, \quad \text{for } N_v = 1, 2, \dots, \quad (47)$$

where H_{sL} has the meaning of the superheating field of the Meissner state ($N_v = 0$) for $L < \infty$ ($H_{sL} > H_s$ for $L < \infty$, and $H_{s\infty} \equiv H_s$). The lower bound in (46), (47) is determined by the exact upper bound in (41). At $H = \sup H = H_{N_v}$, when all $\phi_n(-L) = \inf \phi_n(-L) = -\pi$, there is instability of the saddle-point type (see the end of section III). Note that both the Meissner solution ($N_v = 0$) and the vortex-plane solutions ($N_v \geq 1$) automatically satisfy the symmetry relations (5).^{3,17}

It is instructive to verify the general inequalities (30), (31). Mathematically, it is sufficient to do this only for H equal to the lower bounds in (46), (47): By continuity arguments, the result will be valid in the whole field range. For the Meissner solution the verification is trivial. For $N_v \geq 1$, we employ the exact expression

$$F \left[\bar{\phi}_n, \frac{d\bar{\phi}_n}{dy}; \bar{H} \right] = r(T) \epsilon^2 \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n, m) \int_{-L}^L dy \frac{d\bar{\phi}_n(y)}{dy} \frac{d\bar{\phi}_m(y)}{dy}, \quad (48)$$

where $\bar{H} \equiv \sqrt{H_{N_v-1}^2 - H_s^2}$. As shown in Ref. 3, in this case

$$\left[\frac{d\bar{\phi}_n(y)}{dy} \right]_{\min} = \frac{d\bar{\phi}_n}{dy}(\pm L) = 2\sqrt{H_{N_v-1}^2 - H_s^2}, \quad \text{for all } n. \quad (49)$$

Combining Eqs. (48), (49), we get exactly the lower bound in (31).

As is emphasized in Ref. 3, the obtained solutions are valid for any N , including the cases $N = 2$ (a single junction) and $N = 3$ (a double-junction stack). (For $N = 2, 3$, we have derived in Ref. 3 exact, closed-form analytical expressions.) The solutions with $N_v \geq 1$ are pure solitons only at $H = \sqrt{H_{N_v-1}^2 - H_s^2}$, when $j_n(\pm L) = \sin \phi_n(\pm L) = 0$ [$j_n(y)$ is the density of the Josephson current³ for $f_n = 1$]. In the rest of the regions (47), we have solitons "dressed" by the Meissner field. In the case of $N = 2$ (the single junction), Owen and Scalapino³⁴ called these regions the " N_v to $N_v + 1$ vortex mode". [Because the principle of superposition does not apply to the nonlinear Eqs. (2), the Meissner and the vortex-plane fields cannot be separated from each other.] It is clear that the vortex-plane solutions for $N > 2$ are a direct generalization of ordinary vortices in single junctions.

Of special interest is the overlap of the regions (46), (47) for $N_v = \bar{N}_v$ and $N_v = \bar{N}_v + 1$. As a result, the obtained solutions cover the whole field range $0 \leq H < \infty$, as they should. Mathematically, the overlap is related to the fact that the solution with $N_v = \bar{N}_v$ cannot be continuously transformed into the solution with $N_v = \bar{N}_v + 1$ by changing H , as is always the case for solitons. For the single junction, the overlap was first established numerically in Ref. 34 and discussed qualitatively in Ref. 35. The overlap practically vanishes for $H_{N_v} \gg H_s$. Given that all H_{N_v} decrease when $W = 2L$ increases,³ the overlap is stronger for large W and can involve several neighboring states. Physically, the actual equilibrium state is the one that corresponds to the absolute minimum of the Gibbs free energy for given H . The rest of the allowed states are metastable. In view of the above-mentioned discontinuity, a transition from the state with $N_v = \bar{N}_v$ to the state with lower Gibbs free energy $N_v = \bar{N}_v + 1$ will necessarily be a phase transition of the first-order type.^{1,2,35} In particular, the lower critical field H_{c1} is determined from the requirement that the Gibbs free energy of the state $N_v = 1$ be equal to that of the Meissner state ($N_v = 0$) and satisfies the relation $\sqrt{H_{sL}^2 - H_s^2} < H_{c1} < H_{sL}$.

C. Topological considerations and stability in the dynamic regime

The stability of the Meissner solution and vortex-plane solitons can be better understood, if we analyze the obtained results from the general point of view of the stability of topological defects in continuum media.^{4,5,6,7,8,9,29} To this end, we consider the density of the Gibbs free energy (12) at the boundaries $y = \pm L$.

Because of the general symmetry relations (6), valid for any solution to (6), (10), the density of the Josephson energy is equal at $y = -L$ and $y = +L$:

$$1 - \cos \phi_n(-L) = 1 - \cos \phi_n(+L), \quad n - 1 \leq x < n, \quad n = 1, 2, \dots, N - 1. \quad (50)$$

Taking into account (11), we conclude that also the density of the total free energy is equal at the boundaries $y = -L$ and $y = +L$ and thus corresponds to the degenerate equilibrium ("vacuum") state, unperturbed by topological defects (solitons). Mathematically, the boundary of the interval $-L \leq y \leq L$ can be considered as a 0-dimensional sphere: $S^0 = \{-L, +L\}$. Given that configurations ϕ_n and $\phi_n + 2\pi Z_n$ ($Z_n = 0, \pm 1, \pm 2, \dots$) are physically indistinguishable, we can fix the values $\phi_n(-L)$ as in (41) and regard the functions

$$\psi_n(+L) \equiv \frac{\phi_n(+L) + \phi_n(-L)}{2\pi} = Z_n \quad (51)$$

as continuous maps of the boundary into the additive group of the integers, \mathbf{Z} : $S^0 \xrightarrow{\psi_n} \mathbf{Z}$. (\mathbf{Z} is the group of the degeneracy of the equilibrium state, or the order-parameter space.) The fact of the existence of topologically nontrivial maps of this type, realized by soliton solutions, is often expressed in terms of the "zeroth homotopy group"^{4,5,6,7,8,9,29} $\pi_0(M)$, where the index "0" stands for the boundary S^0 and M is the order-parameter space:

$$\pi_0(\mathbf{Z}) = \mathbf{Z}. \quad (52)$$

[Note that $\pi_0(M)$ is merely the set of disconnected components of the space M .] Because of the boundary conditions (4), all ψ_n at $H > 0$ realize the same mapping: $Z_1 = Z_2 = \dots = Z_{N-1} \equiv Z$. The external field $H > 0$ breaks the symmetry $\phi_n \rightarrow -\phi_n$ [see the second term in (12)]. Therefore, only the values $Z \equiv N_v = 0, 1, 2, \dots$ are allowed, with $N_v = 0$ being the "vacuum", Meissner state. In this way, we arrive at the natural topological classification (45) of the

minimizers of (12). Owing to the continuity conditions (36), (37), variation in (17), (18) is allowed only with respect to trial functions $\{\phi_n\}$ that have the same end points $\phi_n(\pm L)$ and the middle point $\phi_n(0)$ as the minimizer $\{\bar{\phi}_n\}$, i.e., $\{\phi_n\}$ are homotopic to $\{\bar{\phi}_n\}$ and belong to the same class (45): hence the stability of $\{\bar{\phi}_n\}$ against continuous perturbations.

Numerical simulations for time-dependent coupled SG equations have revealed exceptional stability of vortex planes in the dynamic regime as well: see figure 7 in Ref. 30, figure 8 in Ref. 31, and figure 3 in Ref. 32. (The authors of Refs. 30,31,32 employ the terms "coherent", "in-phase" or "phase-locked modes" instead of our term "vortex planes" that we prefer for physical reasons.) Although a detailed analysis of the dynamics of vortex planes is beyond the scope of this paper and will be done elsewhere, the results of Refs. 30,31,32 can be explained already at this stage.

In the absence of dissipation, the dynamic SG equations, describing an evolution of the system in the time interval $t_i \leq t \leq t_f$, can be derived from a corresponding Lagrangian by use of a variational principle. The requirement that the surface variation vanish on the whole perimeter of the space-time boundary leads to the conditions $\delta\phi_n(y, t_i) = \delta\phi_n(y, t_f) = 0$ and a generalization of the conservation law for the Josephson flux Φ_J , Eq. (23):

$$\Phi_J = \sum_{n=1}^{N-1} \Phi_n \frac{1}{2\pi} \int_{-L}^L dy \frac{\partial\phi_n(y, t)}{\partial y} = \sum_{n=1}^{N-1} \Phi_n \frac{\phi_n(L, t) - \phi_n(-L, t)}{2\pi} = \text{const}, \quad (53)$$

which means that the differences $\phi_n(L, t) - \phi_n(-L, t)$ do not depend on t . Thus, by fixing the boundary conditions $\phi_n(\pm L, t)$ at $t = t_i$ as in subsection IV.A, we fix the initial value of the flux $\Phi_J = \bar{\Phi}_J$ that will not change in the course of the evolution of the system from $t = t_i$ to $t = t_f$. The topological type of the solution [see (46)] will not change, either:

$$N_v = \left[\frac{1}{2\pi} \int_{-L}^L dy \frac{\partial\phi_n(y, t)}{\partial y} \right] = \left[\frac{\phi_n(L, t) - \phi_n(-L, t)}{2\pi} \right] = \text{const}. \quad (54)$$

As usual,^{5,6,7,8,9} this situation can be formalized in terms of the conserved topological current

$$j_\mu = \sum_{n=1}^{N-1} \frac{\Phi_n}{2\pi} \epsilon_{\mu\tau} \partial_\tau \phi_n, \quad \partial_\mu j_\mu = 0, \quad (55)$$

where $\mu, \gamma = 0, 1$; $\partial_\mu = (\partial_t, \partial_x)$; and $\epsilon_{\mu\tau}$ is the antisymmetric symbol on two indices, $\epsilon_{01} = -\epsilon_{10} = 1$; with $\Phi_J = \int_{-L}^L dy j_0$ being the topological charge.

As should be clear from these results, time-dependent SG equations alone cannot describe Josephson-vortex penetration, i.e., an evolution of the system from the topologically trivial Meissner state, $N_v = 0$, to a state with $N_v \neq 0$. Unfortunately, this important issue has not been realized in Ref. 12 that claims to have "demonstrated a dynamic process of vortex penetration" by means of numerical simulations for time-dependent SG equations.

V. DISCUSSION

A. Unstable solutions to the SG equations

As is proved in section III, all non-topological, non-soliton solutions to (2), (10) that do not meet the requirement of the flux conservation (23) are absolutely unstable: They are nothing but saddle points of the Gibbs free-energy functional (12), cannot be assigned any "free energy" and are therefore unobservable. Since the requirement of the continuity of variations in (17) does not impose on such solutions any constraints of the type (36), (37), they can be continuously transformed into the stable Meissner solution or a vortex-plane solution, representing the actual minimum of (12) at a given H , by a series of infinitesimal deformations of ϕ_n without a violation of the boundary conditions (10). [A clear illustration of such a transformation for non-topological defects in a system of planar spins see in Ref. 29, section II.B.]

Analytically, all unstable configurations for $H \geq 0$ can be obtained using the symmetry relations (6), as solutions to the boundary value problem

$$\frac{d\phi_n}{dy}(-L) = 2H, \quad \phi_n(0) = \pi Z_n, \quad Z_n = 0, 1, 2, \dots \quad (56)$$

that violate topological boundary conditions, derived in subsection IV.A. For example, one can set in (56) $Z_1 = Z_2 = \dots = Z_{N-1} \equiv Z$ and increase H beyond the upper bound H_{N_v} of the stability regions (46), (47) for a given $N_v = Z$. By continuously increasing H beyond the stability region of the Meissner state, unstable configurations with $Z = 0$, interpreted as "Josephson-vortex penetration", were obtained in numerical simulations for static SG equations (Ref. 31, figure 7) and time-dependent SG equations (Ref. 12, figure 2). Analogous instability for $Z \geq 1$ is demonstrated by numerical simulations in the dynamic regime [$H = \text{const} > 0$, and an increasing transport current $I > I_c(H)$] in Ref. 32: see region IV in figure 3 therein; region III corresponds to dynamically stable vortex planes.

Unstable solutions appear also when not all Z_n in (56) are equal to each other. Thus, an unstable "single Josephson vortex" corresponds to the choice $Z_l = 1$, $Z_{n \neq l} = 0$. Solutions of this type were obtained in several numerical simulations.^{16,25,30} By way of illustration, we consider here only the case $H = 0$. As is explained at the end of subsection IV.A, the only stable configuration for $H = 0$, $L < \infty$ is the trivial Meissner state $\phi_1 = \phi_2 = \dots = \phi_{N-1} \equiv 0$. Figure 5 in Ref. 30, and figures 1, 2 in Ref. 16 clearly show that the solutions presented therein, in reality, are characterized by all $Z_n = \left[\frac{1}{2\pi} \int_{-L}^L dy \frac{d\phi_n(y)}{dy} \right] = 0$ and, thus, belong to the class $\mathbf{Q}_v = (0, 0, \dots, 0)$ of the general topological classification (45). By means of continuous deformations, they can be transformed into the trivial Meissner solution.

Other unstable solutions, available in the literature, can be analyzed along the same lines. In particular, the "triangular Josephson-vortex lattice with the period $x_p = 1$ ", proposed in Ref. 19, corresponds to the case $Z_{\text{odd}} = Z$, $Z_{\text{even}} = Z + 1$.

B. A comparison with Abrikosov vortices in type-II superconductors

As the formation of a vortex-plane soliton involves only phase differences between successive S-layers, it does not affect the topology of the layered superconductor. In contrast, the appearance of a linear (\mathbf{R}^1) singularity of the order parameter $\Delta(\mathbf{r}) = |\Delta(\mathbf{r})| \exp[i\varphi(\mathbf{r})]$ is necessary for the formation of an Abrikosov vortex in continuum type-II superconductors. Thus, in the presence of a single Abrikosov vortex, the topology of the continuum type-II superconductor changes from \mathbf{R}^3 (the three-dimensional Euclidean space) to $\mathbf{R}^3/\mathbf{R}^1 = S^1$. Therefore, the notion of the "vortex core"¹⁴ is inherent (both physically and mathematically) to the Abrikosov vortex and is meaningless in the case of the vortex plane.

An isolated Abrikosov vortex is itself a stable object, both topologically and energetically. (The latter can be proved by the use of the same mathematical methods as those employed in our section III: see, e.g., Refs. 7,8,9.) Therefore, an equilibrium state of N_v Abrikosov vortices is determined by comparing the values of the Ginzburg-Landau free-energy functional for different spatial configurations, which yields the well-known triangular lattice as the most favorable one.¹⁴ In contrast, the notion of the "Josephson-vortex lattice" is senseless for layered superconductors: One can only speak of N_v -soliton (vortex-plane) states, with $N_v = 0$ representing the Meissner state, and each vortex plane being a "Josephson vortex" itself.

In the case of extreme type-II superconductors, the linear singularities, associated with Abrikosov vortices, can be easily incorporated into the framework of the simple London model.¹⁴ The resulting equation is a linear inhomogeneous partial differential equation for the local field. Owing to linearity, the local field is a superposition of the Meissner and vortex fields. As is emphasized in subsection IV.B, this is not the case for layered superconductors because of the nonlinearity of the SG equations (2). Unfortunately, this important issue was not understood in some publications concerned with Josephson-vortex penetration.³⁶

As we can see, there is no "analogy" between the Abrikosov-vortex structure in continuum type-II superconductors and the Josephson-vortex structure in layered superconductors in the naive sense.¹¹ Instead, there is a much subtler mathematical analogy: The topological classification of vortex configurations in type-II superconductors is isomorphic to that in layered superconductors. A proof is straightforward. For the reasons explained above, the boundary of a type-II superconductor is, in general, topologically equivalent to a one-dimensional sphere (a circle) S^1 . The order parameter space is $M = U(1)$ (the symmetry group of quantum electrodynamics). Topologically, $U(1) = S^1$. Thus, soliton solutions, in this case, realize nontrivial maps $S^1 \rightarrow S^1$. All the continuous maps $S^1 \rightarrow S^1$ have a group structure of the fundamental (or first homotopy) group $\pi_1(S^1)$.^{4,5,6,7,8,9,29} Given that $S^1 = \mathbf{R}/\mathbf{Z}$ (\mathbf{R} is the additive group of the real numbers), we can write

$$\pi_1(S^1) = \pi_1(\mathbf{R}/\mathbf{Z}) = \pi_0(\mathbf{Z}) = \mathbf{Z}, \quad (57)$$

which should be compared with (52). As in the case of layered superconductors, the external magnetic field $H > 0$ breaks the symmetry $\varphi \rightarrow -\varphi$. Thus, only the states parameterized by $N_v = 0, 1, \dots$ are possible, with $N_v = 0$ being the "vacuum", Meissner state.

To conclude this discussion, we have to clarify a typical misunderstanding¹² concerning the role of the laminar model¹⁴ in type-II superconductivity. In reality, the order-parameter space of a continuum type-II superconductor, $M = S^1$, precludes the existence of topologically stable plane defects, envisaged by the laminar model. Indeed, consider two points $P_1 = (x_0, a, z_0)$, $P_2 = (x_0, b, z_0)$ on the opposite sides of such a defect, in unperturbed regions of the superconductor. Join these points by a continuous path, parameterized by $a \leq y \leq b$. The boundary of the interval $[a, b]$ is a 0-dimensional sphere $S^0 = \{a, b\}$, which leads us to a consideration of the maps $S^0 \rightarrow S^1$. However, the pertinent homotopy group $\pi_0(S^1)$ is trivial,^{4,5,6,8,9,29} i.e.,

$$\pi_0(S^1) = 0, \quad (58)$$

in contrast to (52) and (57). Topological instability of the "laminar solution" hardly allows one to expect that this solution corresponds to any local minimum of the Ginzburg-Landau free-energy functional. Therefore, a comparison with unstable "isolated fluxons" or "triangular Josephson-vortex lattices" in layered superconductors is much more appropriate than the far-fetched¹² "similarity" to the vortex plane.

C. The interpretation of experimental data

Experimental observations of the vortex structure in layered superconductors can be roughly subdivided into two groups: (i) direct observations, allowing one to "visualize" the flux distribution;^{23,24,37} (ii) indirect observations (i.e., measurements of c -axis transport properties,^{25,38,39} magnetization,⁴⁰ and the upper critical field⁴¹). Here, we present an overview of these observations, showing that all the experimental data available up to now can be explained in terms of the stable vortex-plane configurations. A detailed quantitative analysis can be done with the use of the results of Refs. 1,2,3.

Josephson-flux distribution, characteristic of vortex planes for $H > 0$, was directly observed on artificial low- T_c stacked junctions in Ref. 23 (by low-temperature scanning electron microscopy) and in Ref. 24 (by polarized neutron reflection). In particular, the double-junction stack²³ ($N = 3$) has revealed the phase-difference symmetry $\phi_1 = \phi_2$, exactly as could be expected from the general relations (5) for vortex planes. In both the experiments,^{23,24} penetration of the flux occurred simultaneously and coherently into all the junctions, in full agreement with the scenario for the vortex planes.^{1,2} Moreover, accompanying measurements of magnetization in Ref. 24 have shown typical oscillations and hysteresis. The oscillations should be viewed as a manifestation of a series of first-order phase transitions, discussed in section IV.2 and Refs. 1,2, whereas the hysteresis is implied by the overlap of the regions (46), (47).

We draw attention to a possible application to high- T_c superconductivity. Oscillations of magnetization in parallel fields, interpreted as evidence of Josephson nature of the flux, have been reported for YBCO in Ref. 40. According to Ref. 40, "the temperature dependence of the magnetization contradicts the present theoretical expectations".

As shown by our self-consistent calculations,^{1,2} the oscillating behavior of the critical Josephson current $I_c(H)$ (the Fraunhofer pattern) is a result of successive penetration of vortex planes and their pinning by the edges of the superconductor. Oscillating $I_c(H)$ dependencies have been observed both on artificial low- T_c stacked junctions^{25,38} and high- T_c layered superconductors BSCCO.^{25,39} "Irregularities" of the dependence $I_c(H)$,^{25,38} such as, e.g., multivaluedness and aperiodicity, can be easily explained by the overlap of the regions (46), (47). Behavior of this type was observed a long time ago on the single Josephson junction,⁴² which confirmed the theoretical prediction of the overlap for ordinary Josephson vortices.³⁴

The most "ancient" experimental confirmation of the stability of vortex planes is provided by observations of the "crossover" behavior of $H_{c2}(T)$ in artificial low- T_c stacked junctions.⁴¹ For $H \preceq H_{c2}$, the condition $f_n = 1$, employed in the derivation of Eqs. (2), is no longer valid. However, periodic modulations of $f_n(y)$, caused by the presence of vortex planes and therefore identical in all the S-layers, account for the observed behavior of $H_{c2}(T)$.^{1,2}

Finally, we want to comment on direct observations of non-equilibrium isolated vortices in layered high- T_c superconductors at $H = 0$.³⁷ As is explained at the end of subsection IV.A, nontrivial flux configurations cannot exist in a layered superconductor with ideal periodicity at $H = 0$, $L < \infty$. However, the presence of structural defects (e.g., stacking faults, as is hinted in Ref. 37) violates the condition of ideal periodicity and should stabilize energetically an otherwise unstable configuration. In this situation, we indeed expect to obtain non-equilibrium isolated vortices, because their self-energy is lower and the c -axis extent is smaller than those of vortex planes at $H = 0$, $L < \infty$.² A detailed mathematical analysis of this case can be done on the basis of the results of Ref. 3.

VI. CONCLUSIONS

In brief, we have solved the problem of the classification of all solutions to (2), (10) with respect to their stability. In our consideration, we have employed exact methods of the calculus of variations, soliton physics and the exact results

of Ref. 3 [the expression for the Gibbs free-energy functional (12), the first integral (11) and the solution to the soliton boundary value problem (2), (4), (44)]. In view of obvious mathematical complexity, the problem of the stability of vortex configurations could not be solved by use of inadequate methods, employed, e.g., in Refs. 11,12,18,19,20,21, and was not even posed in any of these publications.

In full agreement with the fundamentals of soliton physics, we have proved that the only minimizers of both the Gibbs and Helmholtz free energy functionals are the Meissner solution (the "vacuum" state) and soliton vortex-plane solutions. They represent the actual equilibrium field configurations. The obtained results allowed us to explain exceptional stability of vortex planes, established in numerical simulations, and to provide a unified interpretation of the experimental data available up to now.

In contrast, non-soliton configurations ("isolated fluxons", triangular Josephson-vortex lattices", etc.), proposed in previous publications, turned out to be absolutely unstable and unobservable: They are nothing but saddle points of the Gibbs free-energy functional and are not even stationary points of the Helmholtz free-energy functional. Physically, these configurations violate conservation laws for the flux and the current.

One may ask a natural question why exactly the unstable configurations were previously proposed as the "equilibrium state", whereas the actual minimizers of the free energy (vortex-plane solitons) were neglected. We think that the answer lies in the following:

i) the hypothesis of an "analogy" with the Abrikosov-vortex structure in type-II superconductors, accepted without any mathematical justification;¹¹

ii) the absence of an exact mathematical definition of the "Josephson vortex". In Refs. 11,18, "isolated Josephson vortices" were discussed without any consideration of the SG equations. The fact that Josephson vortices are nothing but static solitons of the SG equations was not realized in subsequent publications, either. For example, the existence of Josephson vortices in the case $W \ll 2\lambda_J$ was denied in Refs. 15,16, which is refuted by our Eqs. (47), valid for any W ;

iii) obvious mathematical mistakes in the treatment of the Lawrence-Doniach model⁴³ for infinite ($N = \infty$) layered superconductors. The neglect^{15,19,20,21} of the surface variation in the variational principle for the Lawrence-Doniach functional resulted in a loss of the conservation laws for the current and the flux, as was first pointed out in Ref. 1,2: Soliton solutions are a corollary of these conservation laws;

iv) the absence of any investigation of analytical properties of the coupled static SG equations for $H > 0$, $W < \infty$. Pertinent soliton solutions were obtained in our papers: in the exactly solvable cases $N = \infty$ (Refs. 1,2), $N = 2, 3$ (Refs. 3,17), and in the general case $2 \leq N < \infty$ (Ref. 3). Standard methods of soliton physics^{4,5,6,7,8,9,28} as well as advanced methods of the calculus of variations and of the theory of differential equations, employed in our analysis, were completely disregarded in previous theoretical publications;^{15,19,20,21}

v) the absence of any attempts to analyze the stability of the proposed "vortex" configurations, both in theoretical publications^{11,18,19,20,21} and numerical simulations.¹⁶ Dynamic stability of vortex planes, noticed in numerical simulations,^{30,31,32} was not understood and neglected;

vi) the neglect of direct experimental observations^{23,24} of the Josephson-vortex structure at $H > 0$: These observations have clearly revealed that exactly the vortex planes (not "isolated fluxons" or "triangular lattices") are the actual equilibrium field configurations;

vii) long-term domination of the subjective point of view¹¹ and the absence of any pluralism of opinion. As a result, the critical remarks²² are neglected, whereas the attempts to clarify the situation within the framework of a rigorous mathematical approach^{1,2} are immediately attacked¹² with the use of inappropriate methods.¹³

We hope that this paper will finally convince both theorists and experimentalists, specializing in the field of weak superconductivity, of the necessity to give up the old, unsound theoretical prejudices: The wealth of magnetic properties of layered superconductors (both low- and high- T_c) cannot be understood without the solitons. One should also think of possible practical applications of the vortex-plane solitons, e.g., in submillimeter-wave generators, as is proposed in Ref. 32. Given the role of the *single* SG equation in different fields of physics (quantum optics;⁷ the Skyrme and the Thirring models in elementary particle physics;^{5,6,7} the theory of dislocations and magnetism, let alone the Josephson effect, in condensed matter physics,⁷ etc.), we expect that our exact results for the *coupled* SG equations (including the *single* one as a particular case) may find applications in these fields as well.

The coupled SG equations for $H > 0$, $W < \infty$ have not been studied in mathematical literature, either. Our exact analytical results for the *static* case constitute only the first step in this direction. The next stage should be analytical properties of *time-dependent* equations. Our paper may stimulate interest in this problem of specialists in applied mathematics as well.

APPENDIX A: THE PROPERTIES OF THE MATRIX $G(n, m)$

The explicit form of $G(n, m)$ is

$$G(n, m) = \frac{1}{2\epsilon\sqrt{1 + \frac{\epsilon^2}{4}}} \left[\mu^{|n-m|} - \frac{\mu^n (\mu^{m-N} - \mu^{N-m}) + \mu^{N-n} (\mu^{-m} - \mu^m)}{\mu^{-N} - \mu^N} \right], \quad (\text{A1})$$

where

$$\mu = 1 + \frac{\epsilon^2}{2} - \epsilon\sqrt{1 + \frac{\epsilon^2}{4}}. \quad (\text{A2})$$

The following properties of $G(n, m)$ are obvious:

$$G(n, m) = G(m, n), \quad (\text{A3})$$

$$G(n, N - m) = G(N - n, m). \quad (\text{A4})$$

The matrix $G(n, m)$ is positive definite, since all its eigenvalues e_j are positive:

$$e_j = \frac{\lambda_j^2}{\epsilon^2}, \quad \lambda_j = \frac{\epsilon}{\sqrt{2 + \epsilon^2 - 2 \cos \frac{\pi j}{N}}}, \quad j = 1, 2, \dots, N - 1. \quad (\text{A5})$$

Of importance are the summation rules:

$$\begin{aligned} \sum_{m=1}^{N-1} G(n, m) &= \frac{1}{\epsilon^2} [1 - G(n, 1) - G(n, N - 1)] \\ &= \frac{1}{\epsilon^2} \left[1 - \frac{\mu^{-n} + \mu^{-N+n} - \mu^n - \mu^{N-n}}{\mu^{-N} - \mu^N} \right], \quad 1 \leq n \leq N - 1; \end{aligned} \quad (\text{A6})$$

$$\sum_{n=1}^{N-1} \sum_{m=1}^{N-1} G(n, m) = \frac{1}{\epsilon^2} \left[N - 1 - \frac{2\sqrt{1 + \frac{\epsilon^2}{4}} - \epsilon}{\epsilon} \frac{1 - \mu^{N-1}}{1 + \mu^N} \right] \equiv \frac{N - 1}{\epsilon^2 H_s^2}, \quad (\text{A7})$$

where

$$\begin{aligned} H_s &= \left[1 - \frac{2\sqrt{1 + \frac{\epsilon^2}{4}} - \epsilon}{\epsilon(N - 1)} \frac{1 - \mu^{N-1}}{1 + \mu^N} \right]^{-\frac{1}{2}} \\ &\equiv \sqrt{\frac{(N - 1)N}{2}} \left[\sum_{k=0}^{[\frac{N}{2}] - 1} \lambda_{2k+1}^2 \cos^2 \frac{\pi(2k + 1)}{2N} \right]^{-\frac{1}{2}} \end{aligned} \quad (\text{A8})$$

is the superheating (penetration) field of a semiinfinite ($0 \leq y < \infty$) layered superconductor.³

APPENDIX B: A RELATIONSHIP TO THE VARIATIONAL PRINCIPLE FOR INFINITE LAYERED SUPERCONDUCTORS ($N = \infty$)

The intralayer currents for $f_n = 1$ are given by³

$$J_n(y) = \frac{d\varphi_n(y)}{dy} - 2A(n, y)$$

$$= \frac{1}{4\pi} [H_n(y) - H_{n+1}(y)], \quad n = 0, 1, \dots, N-1, \quad (\text{B1})$$

where $H_0(y) = H_N(y) \equiv H$. Using the second relation in (B1), we get

$$\sum_{n=0}^{N-1} J_n(y) = 0, \quad (\text{B2})$$

which is the conservation law for the total intralayer current. Moreover, in view of (5) and (A3), (A4),

$$H_n(y) = H_{N-n}(y). \quad (\text{B3})$$

Hence,

$$J_n(y) = -J_{N-n-1}(y). \quad (\text{B4})$$

Using the first relation in (B1), we write

$$\begin{aligned} & \int_{-L}^L dy [J_n(y) - J_{n-1}(y)] \\ &= \phi_n(L) - \phi_n(-L) - \int_{-L}^L dy [A(n, y) - A(n-1, y)], \quad n = 1, 2, \dots, N-1. \end{aligned} \quad (\text{B5})$$

The second term on the right-hand side of (B5) is the flux between the S-layers n and $n-1$. We can therefore rewrite (B5) using (8):

$$\begin{aligned} & \int_{-L}^L dy [J_n(y) - J_{n-1}(y)] = HW [G(n, 1) + G(n, N-1)] \\ & + \phi_n(L) - \phi_n(-L) + \frac{\epsilon^2}{2} \sum_{m=1}^{N-1} G(n, m) [\phi_m(L) - \phi_m(-L)], \quad n = 1, 2, \dots, N-1. \end{aligned} \quad (\text{B6})$$

In view of the flux-conservation conditions (24), the variation of the right-hand side of (B6) vanishes, hence

$$\delta J_n(y) = \delta J_{n-1}(y), \quad n = 1, 2, \dots, N-1. \quad (\text{B7})$$

Combined with the current-conservation law (B2), relations (B7) yield:

$$\delta J_n(y) = 0, \quad n = 0, 1, \dots, N-1, \quad (\text{B8})$$

which means that partial intralayer currents are also conserved.

Consider the case $N \gg 1$. For n satisfying the condition $[\epsilon^{-1}] \ll n \ll N-1 - [\epsilon^{-1}]$, we can proceed to the limit $N \rightarrow \infty$ in the second relation (B1), obtaining

$$J_n(y) = \frac{d\varphi_n(y)}{dy} - 2A(n, y) = 0. \quad (\text{B9})$$

This is exactly the result derived for the infinite ($N = \infty$) layered superconductor in Refs. 1,2 by means of an exact variational principle, based on the use of the conservation law for the total intralayer current.

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